By,

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B.Sc Sem-6 MIC

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Que :-

Baire's Category Theorem :-

Theorem :-

If $\{A_n\}$ is a Sequence of nowhere dense subsets of a Complete metric space (X,d), then there exists a point in X which is not in any one of the A_n's i.e $X \neq \bigcup_{n=1}^{\infty} A_n$

Proof :- Let X be a Complete metric space

Let $\{A_n\}$ be any sequence of nowhere dense subset of X.

Since X is open being a non empty subset of itself and A_1 is nowhere dense in X then by the Preceding lemma X Contains an open sphere disjoint from A_1 i.e for given $r_1 > 0$ $0 < r_1 < 1$ $\exists$ an open sphere $= S(x_1, r_1)$ in X

Such that $S(x_1, r_1) \cap A_1 = \phi$

Let F_1 be the Concentric closed Sphere of radius $\frac{1}{2} r_1$ i.e $F_1 = S\left[x_1, \frac{r_1}{2}\right]$

Then Consider the interior of F_1

Clearly $\text{int}(F_1) = F_1^0$ is a non empty open set

Let $x_2 \in \text{int}(F_1)$

Since A_2 is nowhere dense, it follows from the lemma that $\text{int}(F_1)$ Contains an open Sphere $S(x_2, r_2)$ of radius $r_2 < \frac{1}{2}$ such that $S(x_2, r_2) \cap A_2 = \phi$

Let F_2 be the concentric closed sphere of radius $\frac{1}{2} r_2$ i.e. $F_2 = S\left[x_2, \frac{r_2}{2}\right]$

Let $x_3 \in \text{int}(F_2)$

Clearly $\text{int}(F_2)$ is a non empty open set

Since A_3 is nowhere dense it follows again from the lemma that $\text{int}(F_3)$ Contains an open sphere $S(x_3, r_3)$ Centred at $x_3 \in \text{int}(F_2)$ and radius $r_3 < \frac{1}{3}$ such that

$$S(x_3, r_3) \cap A_3 = \phi$$

Let F_3 be the Concentric Closed Sphere of radius $\frac{1}{2} r_3$

Continuing in this manner we get a decreasing Sequence $[F_n]$ of non empty closed Subset of X ($\because$ each Closed Sphere is a Closed Set)

where $F_n = S\left[x_n, \frac{1}{2} r_n\right]$

and $F_{n+1} = S\left[x_{n+1}, \frac{1}{2} r_{n+1}\right] \subset S(x_{n+1}, r_{n+1}) \subseteq S\left[x_n, \frac{1}{2} r_n\right] = F_n$

i.e $F_{n+1} \subseteq F_n \ \forall n$

Also $\delta(F_n) \to 0$ as $n \to \infty$

in this manner we Construct a nested Sequence $[F_n]$ of Closed Spheres having the following two properties

(i) For each positive integer F_n does not interest $A_1 \text{--} A_2 \text{--} A_n$

(ii) $\delta(F_n) \to 0$ as $n \to \infty$

Since X is given to be Complete therefore by Cantor's intersection theorem, we Conclude that $\bigcap_{n=1}^{\infty} F_n$ Contains exactly one point

Say $x \in X$

$\Rightarrow x \in F_n \ \forall n$

$\Rightarrow x \in S\left[x_n, \frac{1}{2} r_n\right] \subset S(x_n, r_n) \ \forall n$

But $S(x_n, r_n) \cap A_n = \phi$

$\Rightarrow x \notin A_n \ \forall n$

$\Rightarrow x \notin \bigcup_{n=1}^{\infty} A_n$

Hence $\bigcup_{n=1}^{\infty} A_n \neq X$.